Quiz 1 **Name:** (Solutions)
Date: 02/07/2024
ESE 531: Spring 2024
Instructor: Yousef El-Laham

Instructions:
Please answer each part in the question below. You have 15 minutes.

Question 1. **(5 points)** Consider a random sample $X_1, \ldots, X_n$ where $X_i \overset{i.i.d.}{\sim} \text{Exp}(\lambda)$, that is

$$p(x; \lambda) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

The population mean and population variance are given by $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathbb{V}[X] = \frac{1}{\lambda^2}$, respectively.

(a) **(2 points)** Consider the statistic $\overline{X_n^2} = \frac{1}{n}\sum_{i=1}^n X_i^2$. Find $\mathbb{E}[\overline{X_n^2}]$.

$$\mathbb{E}[\overline{X_n^2}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i^2\right]$$

$$= \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X_i^2] \quad \leftarrow \text{by linearity}$$

$$= \frac{1}{n}\sum_{i=1}^n \mathbb{E}[X^2] \quad \leftarrow \text{identically distributed}$$

$$= \mathbb{E}[X^2] = \mathbb{V}[X] + \mathbb{E}[X]^2 = \frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2 = \boxed{\frac{2}{\lambda^2}}$$

(b) **(1 point)** Is the statistic $\overline{X_n^2}$ unbiased? If not, what is the bias?

Yes, as an intermediate step, we showed $\mathbb{E}[\overline{X_n^2}] = \mathbb{E}[X^2]$

(c) **(2 points)** The weak law of large numbers (WLLN) tells us that the sample mean $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ converges in probability to the population mean, that is $\bar{X}_n \xrightarrow{p} \mathbb{E}[X]$. Is it true that $\overline{X_n^2} \xrightarrow{p} \mathbb{E}[X^2]$. Answer **(Yes/No)** and provide a justification as to why.

Yes. Consider the transformation $Y_i = X_i^2$ for all $i = 1, \ldots, n$. Clearly $\{Y_i\}_{i=1}^n$ are a random sample ~~to~~ from the population $p(y)$ (exact distribution can be found by change of variables formula). By WLLN, as long as $\mathbb{E}[Y_i]$ exists and $\mathbb{V}[Y_i] < \infty$, then the sample mean $\frac{1}{n}\sum_{i=1}^n Y_i \xrightarrow{p} \mathbb{E}[Y_i] = \mathbb{E}[X^2]$