ESE 531: Statistical Learning and Inference

Homework 2: Method of Moments and Maximum Likelihood Estimators

1. Suppose $X_1, \dots, X_n$ are i.i.d. distributed according to the following (discrete) distribution:

$$p(x;\alpha,\beta) = \begin{cases} 1-\beta-\alpha, & x = 0, \\ \beta, & x = 1, \\ \alpha, & x = 2, \\ 0, & \text{otherwise} \end{cases}$$

   (a) For what values of α and β is $p(x;\alpha,\beta)$ a valid pmf?

   Solution. Let $\mathcal{S}_x = \{0,1,2\}$ denote the support (i.e., the region of nonzero probability) of the distribution $p(x;\alpha,\beta)$. For the pmf to be valid it must satisfy the following properties:

$$p(x;\alpha,\beta) \geq 0, \quad x \in \mathcal{S}_x$$
$$\sum_{x\in\mathcal{S}_x} p(x;\alpha,\beta) = 1$$

   It is evident from analyzing both of the above constraints that that α and β must satisfy $0 \leq \alpha+\beta \leq 1$ with $\alpha \geq 0$ and $\beta \geq 0$.

   (b) Determine the method of moments estimator for α and β.

   Solution. To obtain the MOME, we first need to compute the theoretical values of the population moments based on the parameter estimates. Since there are two unknowns, we utilize the first two moments $g_1(\widehat{\alpha},\widehat{\beta}) = \mathbb{E}_{p(x;\widehat{\alpha},\widehat{\beta})}[X]$ and $g_2(\widehat{\alpha},\widehat{\beta}) = \mathbb{E}_{p(x;\widehat{\alpha},\widehat{\beta})}\left[X^2\right]$. For the first moment, we have:

$$\begin{aligned} g_1(\widehat{\alpha},\widehat{\beta}) &= \mathbb{E}_{p(x;\widehat{\alpha},\widehat{\beta})}[X] \\ &= \sum_{x\in\mathcal{S}_x} x p(x;\widehat{\alpha},\widehat{\beta}) \\ &= 0 \times (1-\widehat{\beta}-\widehat{\alpha}) + 1 \times \widehat{\beta} + 2 \times \widehat{\alpha} \\ &= \widehat{\beta} + 2\widehat{\alpha} \end{aligned}$$

   For the second moment, we have:

$$\begin{aligned} g_2(\widehat{\alpha},\widehat{\beta}) &= \mathbb{E}_{p(x;\widehat{\alpha},\widehat{\beta})}\left[X^2\right] \\ &= \sum_{x\in\mathcal{S}_x} x^2 p(x;\widehat{\alpha},\widehat{\beta}) \\ &= 0^2 \times (1-\widehat{\beta}-\widehat{\alpha}) + 1^2 \times \widehat{\beta} + 2^2 \times \widehat{\alpha} \\ &= \widehat{\beta} + 4\widehat{\alpha} \end{aligned}$$

   MOME estimators are obtained by solving the following system of equations:

$$g_1(\widehat{\alpha},\widehat{\beta}) = \bar{X}_n \implies \widehat{\beta} + 2\widehat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$g_2(\widehat{\alpha},\widehat{\beta}) = \overline{X_n^2} \implies \widehat{\beta} + 4\widehat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

   Subtracting the second equation by the first equation, we have that

$$2\widehat{\alpha} = \overline{X_n^2} - \bar{X}_n$$

and so the MOME of α is

$$\widehat{\alpha} = \frac{\overline{X_n^2} - \bar{X}_n}{2}$$

Taking this solution for $\widehat{\alpha}$ and plugging back into the first equation, we have:

$$\widehat{\beta} + 2\widehat{\alpha} = \bar{X}_n \implies \widehat{\beta} = 2\bar{X}_n - \overline{X_n^2}$$

and so the MOME of β is

$$\widehat{\beta} = 2\bar{X}_n - \overline{X_n^2}$$

(c) Determine if the MOME estimator is biased.

Solution. To determine if an estimator is biased, we need to compute its expected value and compare it to the desired parameter. Bias is defined as $\text{bias}(\widehat{\theta}) = \mathbb{E}[\widehat{\theta}] - \theta$. For this problem, we need to find $\mathbb{E}[\widehat{\alpha}]$ and $\mathbb{E}[\widehat{\beta}]$ to determine the bias. First, we have:

$$\begin{aligned}
\mathbb{E}[\widehat{\alpha}] &= \mathbb{E}\left[\frac{\overline{X_n^2} - \bar{X}_n}{2}\right] \\
&= \frac{1}{2}\left(\mathbb{E}\left[\overline{X_n^2} - \bar{X}_n\right]\right) \\
&= \frac{1}{2}\mathbb{E}\left[\overline{X_n^2}\right] - \frac{1}{2}\mathbb{E}\left[\bar{X}_n\right] \\
&= \frac{1}{2}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] - \frac{1}{2}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \\
&= \frac{1}{2}\mathbb{E}[X^2] - \frac{1}{2}\mathbb{E}[X] \\
&= \frac{1}{2}(\beta + 4\alpha) - \frac{1}{2}(\beta + 2\alpha) \\
&= \alpha
\end{aligned}$$

and so $\mathbb{E}[\widehat{\alpha}] = \alpha$, meaning that our estimator for α is unbiased. For $\widehat{\beta}$, we have:

$$\begin{aligned}
\mathbb{E}[\widehat{\beta}] &= \mathbb{E}[2\bar{X}_n - \overline{X_n^2}] \\
&= 2\mathbb{E}[\bar{X}_n] - \mathbb{E}\left[\overline{X_n^2}\right] \\
&= 2\mathbb{E}[X] - \mathbb{E}[X^2] \\
&= 2(\beta + 2\alpha) - (\beta + 4\alpha) \\
&= \beta
\end{aligned}$$

We have shown that both $\mathbb{E}[\widehat{\alpha}] = \alpha$ and $\mathbb{E}[\widehat{\beta}] = \beta$. Therefore, our MOME estimator is unbiased.

(d) Find the maximum likelihood estimator of α and β.

Solution. To find the MLE, we first need to determine the likelihood function $L(\alpha, \beta; x_1, \ldots, x_n)$, $x_1, \ldots, x_n$ are realizations (e.g., observed data) of the random sampled $X_1, \ldots, X_n$. For this problem, it is helpful to write the population distribution as:

$$p(x_i; \alpha, \beta) = (1 - \beta - \alpha)^{\mathbf{1}(x_i=0)}\beta^{\mathbf{1}(x_i=1)}\alpha^{\mathbf{1}(x_i=2)},$$

where $\mathbf{1}(\cdot)$ denotes the indicator function. Note that this representation of the population distribution is equivalent to the piece-wise representation shown in the problem statement. The likelihood function is defined as:

$$\begin{aligned}
L(\alpha, \beta; x_1, \ldots, x_n) &= p(x_1, \ldots, x_n; \alpha, \beta) \\
&= \prod_{i=1}^{n} p(x_i; \alpha, \beta) \\
&= (1 - \beta - \alpha)^{\sum_{i=1}^{n}\mathbf{1}(x_i=0)}\beta^{\sum_{i=1}^{n}\mathbf{1}(x_i=1)}\alpha^{\sum_{i=1}^{n}\mathbf{1}(x_i=2)}
\end{aligned}$$

Define $n_0 = \sum_{i=1}^n \mathbf{1}(x_i = 0)$, $n_1 = \sum_{i=1}^n \mathbf{1}(x_i = 1)$, and $n_2 = \sum_{i=1}^n \mathbf{1}(x_i = 2)$. We can right down the likelihood function in the following more simple form:

$$L(\alpha, \beta; x_1, \ldots, x_n) = (1 - \beta - \alpha)^{n_0} \beta^{n_1} \alpha^{n_2},$$

since n_0, n_1, and n_2 are just constants. Next, we convert the likelihood into the log-likelihood, since it is easier to differentiate:

$$\begin{aligned}
\ell(\alpha, \beta; x_1, \ldots, x_n) &= \log L(\alpha, \beta; x_1, \ldots, x_n) \\
&= n_0 \log(1 - \beta - \alpha) + n_1 \log \beta + n_2 \log \alpha
\end{aligned}$$

We now can differentiate the log-likelihood with respect to both unknown parameters and set both (partial) derivatives equal to 0:

$$\begin{aligned}
\left.\frac{\partial \ell}{d\alpha}\right|_{\alpha=\widehat{\alpha}, \beta=\widehat{\beta}} &= \frac{-n_0}{1 - \widehat{\beta} - \widehat{\alpha}} + \frac{n_2}{\widehat{\alpha}} = 0 \\
\left.\frac{\partial \ell}{d\beta}\right|_{\alpha=\widehat{\alpha}, \beta=\widehat{\beta}} &= \frac{-n_0}{1 - \widehat{\beta} - \widehat{\alpha}} + \frac{n_1}{\widehat{\beta}} = 0
\end{aligned}$$

Solving this system of equations will yield the MLE $\widehat{\alpha} = \frac{n_2}{n_0 + n_1 + n_2} = \frac{n_2}{n}$ and $\widehat{\beta} = \frac{n_1}{n_0 + n_1 + n_2} = \frac{n_1}{n}$.

2. Consider a random sample $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{U}(0, b)$, where $\mathcal{U}(0, b)$ denotes a uniform random variable with lower limit 0 and upper limit b, i.e.,

$$p(x; b) = \begin{cases} \frac{1}{b}, & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the method of moments estimator of b.

Solution. To find the MOME estimator, we need to first compute the theoretical moments (up to the number of unknowns). Since there is only one unknown, we only need to compute the first moment:

$$\begin{aligned}
g_1(\widehat{b}) &= \mathbb{E}_{p(x;\widehat{b})}[X] \\
&= \int_0^b x p(x; \widehat{b}) dx \\
&= \frac{b}{2}
\end{aligned}$$

Setting this equal to the sample mean, we arrive at the following MOME:

$$\widehat{b} = 2\bar{X}_n$$

(b) Is the method of moments estimator biased? Does it always produce a "valid" estimate?

Solution. To check if it is biased:

$$\begin{aligned}
\mathbb{E}[\widehat{b}] &= \mathbb{E}\left[2\bar{X}_n\right] \\
&= 2\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n X_i\right] \\
&= 2\mathbb{E}[X] \\
&= 2\left(\frac{b}{2}\right) = b
\end{aligned}$$

Since $\mathbb{E}[\widehat{b}] = b$, the estimator is unbiased. To check if the estimator produces valid estimates, we can consider a counter example: $x_1 = 0, x_2 = 0, x_3 = 0$, and $x_4 = 1$. For this set of data, it is easy to check that realized estimate is $\widehat{b} = \frac{1}{2}$. This estimate is not valid since $\widehat{b} < x_4$.

(c) Find the maximum likelihood estimator of b.

Solution. First, we determine the likelihood function. Note, it is important to clearly include constraints. To make our lives easier, we can write the pdf in the following form:

$$p(x; b) = \frac{1}{b} \times \mathbf{1}(0 \leq x \leq b)$$

Here, the indicator function serves as a means to account for the constraint that $x \in [0, b]$, where

$$\mathbf{1}(0 \leq x \leq b) = \begin{cases} 1, & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

Then, the likelihood function can readily be determined as:

$$\begin{aligned} L(b; x_1, \ldots, x_n) &= p(x_1, \ldots, x_n; b) \\ &= \prod_{i=1}^{n} \frac{1}{b} \times \mathbf{1}(0 \leq x_i \leq b) \\ &= \left(\frac{1}{b}\right)^n \prod_{i=1}^{n} \mathbf{1}(0 \leq x_i \leq b) \end{aligned}$$

Normally, our next steps would be to find the log-likelihood and differentiate; however, we cannot do that here. The term $\mathbf{1}(0 \leq x \leq b)$ is not differentiable and so taking the derivative is a futile task. Therefore, we must determine the MLE by observation:

- By the constraints, we know that $\widehat{b} > 0$.
- Consider the case that $\widehat{b} < \max(x_1, \ldots, x_n)$, that is $\widehat{b}$ is less than the maximum value of the observed data. Then, there exists an x_i such that $x_i > \widehat{b}$. Under this choice $\mathbf{1}(0 \leq x_i \leq b) = 0$, which would drive the term $\prod_{i=1}^{n} \mathbf{1}(0 \leq x_i \leq b)$ to 0.
- In the case that $\widehat{b} \geq \max(x_1, \ldots, x_n)$, we can see that the likelihood would simply be $\left(\frac{1}{b}\right)^n$

Therefore, we can write the likelihood function (as a function of b) as the following piece-wise function:

$$L(b; x_1, \ldots, x_n) = \begin{cases} \left(\frac{1}{b}\right)^n, & b \geq \max(x_1, \ldots, x_n) \\ 0, & \text{otherwise} \end{cases}$$

Since the term $\left(\frac{1}{b}\right)^n$ decreases as b gets larger, we find that value of b that maximizes the likelihood is exactly at the maximum of the observed data. Therefore,

$$\widehat{b} = \max(X_1, \ldots, X_n)$$

(d) Is the maximum likelihood estimator biased? (*Hint: Find the cdf of the estimator and then differentiate to obtain the pdf*)

Solution. We need to check if $\mathbb{E}[\widehat{b}] = b$ to check it is unbiased. In this case, to find $\mathbb{E}[\widehat{b}]$, it is easier to first find the pdf of $\widehat{b}$ and then take its expected value. We can obtain the pdf by first finding the cdf and then differentiating. The cdf of $\widehat{b} = \max(X_1, \ldots, X_n)$ is given by:

$$\begin{aligned} F_{\widehat{b}}(y) &= \mathbb{P}(\widehat{b} \leq y) \\ &= \mathbb{P}(\max(X_1, \ldots, X_n) \leq y) \\ &= \mathbb{P}(X_1 \leq y, X_2 \leq y, \ldots, X_n \leq y) \\ &= \prod_{i=1}^{n} \mathbb{P}(X_i \leq y) \\ &= \prod_{i=1}^{n} \mathbb{P}(X \leq y) \\ &= F_X(y)^n = \left(\frac{y}{b}\right)^n \end{aligned}$$

Differentiating with respect to y, we have:

$$f_{\widehat{\beta}}(y) = n\left(\frac{y}{b}\right)^{n-1}\left(\frac{1}{b}\right) = \frac{n}{b^n}y^{n-1}$$

To find the expected value, we have:

$$\begin{aligned}\mathbb{E}[\widehat{b}] &= \int_0^b y \times \frac{n}{b^n}y^{n-1}dy \\ &= \frac{n}{b^n}\int_0^b y^n dy \\ &= \frac{n}{b^n}\left(\frac{y^{n+1}}{n+1}\right)\Bigg|_0^b \\ &= \left(\frac{n}{n+1}\right)b\end{aligned}$$

Therefore, the estimator is biased with bias

$$\text{bias}(\widehat{b}) = \mathbb{E}[\widehat{b}] - b = -\left(\frac{1}{n+1}\right)b$$

Notice that as $n \to \infty$, the bias tends to 0.

3. Let $X_1, \dots, X_n$ be a random sample from the pdf:

$$p(x;\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty$$

   (a) Find the maximum likelihood estimator of θ.

   Solution. To find the MLE, we first write down the likelihood function. This one is tricky since $x \in [\theta, \infty)$ and so we can use the indicator function to help us. Following similar steps to the MLE of a uniform distribution, we arrive at the following likelihood function:

$$L(\theta; x_1, \dots, x_n) = \begin{cases}\theta^n\left(\prod_{i=1}^n \frac{1}{x_i^2}\right), & \theta \leq \min(x_1, \dots, x_n) \\ 0, & \text{otherwise}\end{cases}$$

   Clearly, the maximum value of the likelihood is achieved at the minimum value of the observed data, and therefore $\widehat{\theta} = \min(X_1, \dots, X_n)$.

   (b) Find the method of moments estimator of θ.

   Solution. First, we compute the theoretical mean:

$$\begin{aligned}\mathbb{E}_{p(x;\widehat{\theta})}[X] &= \int_\theta^\infty x \times \theta x^{-2}dx \\ &= \log(x)\Bigg|_\theta^\infty = \log(\infty) - \log(\theta) = \infty\end{aligned}$$

   Clearly, the first moment does not exist and therefore, the method of moments estimator does not exist.

4. Consider the following probabilistic model for a sinusoidal signal corrupted by noise:

$$X_i = A\sin(2\pi\omega t_i + \phi) + \epsilon_i,$$

   where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ and t_i denotes the time at which X_i is sampled.

   (a) Determine the population distribution $p(x;\theta)$, where $\theta = \{A, \omega, \phi, \sigma^2\}$.

Solution. With the model parameters considered fixed, the probabilistic model is simply a Gaussian random variable ϵ_i shifted by $A\sin(2\pi\omega t_i + \phi)$. Therefore, we can equivalently write the population distribution as:

$$X_i \sim p(x_i;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - A\sin(2\pi\omega t_i + \phi))^2}{2\sigma^2}\right)$$

(b) Determine the log-likelihood $\ell(\theta; x_1, \ldots, x_n)$.

Solution. We begin by writing the likelihood function:

$$\begin{aligned}
L(\theta; x_1, \ldots, x_n) &= \prod_{i=1}^{n} p(x_i;\theta) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x_i - A\sin(2\pi\omega t_i + \phi))^2}{2\sigma^2}\right) \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - A\sin(2\pi\omega t_i + \phi))^2\right)
\end{aligned}$$

The log-likelihood is then obtained as:

$$\ell(\theta; x_1, \ldots, x_n) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - A\sin(2\pi\omega t_i + \phi))^2$$

(c) Suppose the amplitude A is unknown, but the remaining parameters ω, ϕ, and σ^2 are assumed known. Find the maximum likelihood estimator of A.

Solution. If all other parameters are assumed to be fixed, we can obtain the MLE of A by taking the partial derivative of ℓ with respect to A and setting it equal to 0:

$$\begin{aligned}
\left.\frac{\partial \ell}{\partial A}\right|_{A=\widehat{A}} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i - \widehat{A}\sin(2\pi\omega t_i + \phi))(-\sin(2\pi\omega t_i + \phi)) \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i \sin(2\pi\omega t_i + \phi) - \widehat{A}\sin^2(2\pi\omega t_i + \phi) = 0 \\
\implies \widehat{A} &= \frac{\sum_{i=1}^{n} X_i \sin(2\pi\omega t_i + \phi)}{\sum_{i=1}^{n} \sin^2(2\pi\omega t_i + \phi)}
\end{aligned}$$

(d) Is the estimator for the amplitude A unbiased?

Solution. We can determine if the estimator is biased by taking the expectation of the estimator:

$$\begin{aligned}
\mathbb{E}[\widehat{A}] &= \mathbb{E}\left[\frac{\sum_{i=1}^{n} X_i \sin(2\pi\omega t_i + \phi)}{\sum_{i=1}^{n} \sin^2(2\pi\omega t_i + \phi)}\right] \\
&= \frac{\sum_{i=1}^{n} \mathbb{E}[X_i] \sin(2\pi\omega t_i + \phi)}{\sum_{i=1}^{n} \sin^2(2\pi\omega t_i + \phi)}
\end{aligned}$$

In this case, we need to be careful, the X_i are independent, but not identically distributed since the mean of X_i depends on the time t_i, which can be different for different X_i. Nevertheless, we can determine $\mathbb{E}[X_i]$ straightforwardly:

$$\mathbb{E}[X_i] = A\sin(2\pi\omega t_i + \phi)$$

Plugging this back in to the equation for $\mathbb{E}[\widehat{A}]$, we arrive the result $\mathbb{E}[\widehat{A}] = A$ and the estimator is unbiased.

(e) Suppose the phase ϕ is unknown, but the remaining parameters A, ω, and σ^2 are assumed known. Write the pseudocode for an algorithm that can find the maximum likelihood estimate of ϕ.

Solution. Taking the partial derivative of the log-likelihood w.r.t. ϕ:

$$\begin{aligned}
\frac{\partial \ell}{\partial \phi} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i - A\sin(2\pi\omega t_i + \phi))(-A\cos(2\pi\omega t_i + \phi)) \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i A\cos(2\pi\omega t_i + \phi) - A^2\sin(2\pi\omega t_i + \phi)\cos(2\pi\omega t_i + \phi) \\
&\propto \sum_{i=1}^{n} x_i\cos(2\pi\omega t_i + \phi) - A\sum_{i=1}^{n}\sin(2\pi\omega t_i + \phi)\cos(2\pi\omega t_i + \phi) \\
&\triangleq g(\phi)
\end{aligned}$$

We can try and set $g(\phi) = 0$ to analytically determine the maximum likelihood, but we will quickly find that this cannot be solved analytically. We can then use the gradient ascent algorithm as a numerical means to solve for the estimate $\widehat{\phi}$. The algorithm is summarized as follows:

- Initialize the estimate using a guess: ϕ_0.
- For $j = 1, 2, \ldots, J$
  - Update the estimate as:
  $$\phi_j = \phi_{j-1} + \eta g(\phi_{j-1})$$
- Return the estimate as $\widehat{\phi} = \phi_J$

5. Consider $X_1, \ldots, X_n$ that is i.i.d. distributed according to a mixture distribution:

$$p(x;\pi) = \pi p_1(x) + (1-\pi)p_2(x), \quad -\infty < x < \infty$$

where $0 \le \pi \le 1$. Let $\mu_1 = \mathbb{E}_{p_1(x)}[X]$ denote the mean of the first mixture component and $\mu_2 = \mathbb{E}_{p_2(x)}[X]$ denote the mean of the second mixture component. Find the method of moments estimator for π and determine the bias of the estimator.

Solution. Since there is only one unknown, we only need to compute the first theoretical moment under the parameter $\widehat{\pi}$:

$$\begin{aligned}
g_1(\widehat{\pi}) &= \mathbb{E}_{p(x;\widehat{\pi})}[X] \\
&= \int_{-\infty}^{\infty} x(\widehat{\pi}p_1(x) + (1-\widehat{\pi})p_2(x))dx \\
&= \widehat{\pi}\int_{\infty}^{\infty} xp_1(x)dx + (1-\widehat{\pi})\int_{-\infty}^{\infty} xp_2(x)dx \\
&= \widehat{\pi}\mu_1 + (1-\widehat{\pi})\mu_2
\end{aligned}$$

Setting this equal to the sample mean, we have:

$$\begin{aligned}
\widehat{\pi}\mu_1 + (1-\widehat{\pi})\mu_2 &= \bar{X}_n \\
\implies \widehat{\pi} &= \frac{\bar{X}_n - \mu_2}{\mu_1 - \mu_2}
\end{aligned}$$

The bias of the estimator can be determine by taking the expectation of the estimator:

$$\begin{aligned}
\mathbb{E}[\widehat{\pi}] &= \mathbb{E}\left[\frac{\bar{X}_n - \mu_2}{\mu_1 - \mu_2}\right] \\
&= \frac{\mathbb{E}[\bar{X}_n] - \mu_2}{\mu_1 - \mu_2} \\
&= \frac{\pi\mu_1 + (1-\pi)\mu_2 - \mu_2}{\mu_1 - \mu_2} \\
&= \pi
\end{aligned}$$

Since $\mathbb{E}[\widehat{\pi}] = \pi$, the esitmator is unbiased.