ESE 531: Statistical Learning and Inference

Homework 2: Method of Moments and Maximum Likelihood Estimators

1. Suppose $X_1, \ldots, X_n$ are i.i.d. distributed according to the following (discrete) distribution:

$$p(x; \alpha, \beta) = \begin{cases} 1 - \beta - \alpha, & x = 0, \\ \beta, & x = 1, \\ \alpha, & x = 2, \\ 0, & \text{otherwise} \end{cases}$$

   (a) For what values of α and β is $p(x; \alpha, \beta)$ a valid pmf?

   (b) Determine the method of moments estimator for α and β.

   (c) Determine if the MOME estimator is biased.

   (d) Find the maximum likelihood estimator of α and β.

2. Consider a random sample $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \mathcal{U}(0, b)$, where $\mathcal{U}(0, b)$ denotes a uniform random variable with lower limit 0 and upper limit b, i.e.,

$$p(x; b) = \begin{cases} \frac{1}{b}, & 0 \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the method of moments estimator of b.

   (b) Is the method of moments estimator biased? Does it always produce a "valid" estimate?

   (c) Find the maximum likelihood estimator of b.

   (d) Is the maximum likelihood estimator biased? (*Hint: Find the cdf of the estimator and then differentiate to obtain the pdf*)

3. Let $X_1, \ldots, X_n$ be a random sample from the pdf:

$$p(x; \theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty$$

   (a) Find the maximum likelihood estimator of θ.

   (b) Find the method of moments estimator of θ.

4. Consider the following probabilistic model for a sinusoidal signal corrupted by noise:

$$X_i = A \sin(2\pi\omega t_i + \phi) + \epsilon_i,$$

   where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ and t_i denotes the time at which X_i is sampled.

   (a) Determine the population distribution $p(x; \theta)$, where $\theta = \{A, \omega, \phi, \sigma^2\}$.

   (b) Determine the log-likelihood $\ell(\theta; x_1, \ldots, x_n)$.

   (c) Suppose the amplitude A is unknown, but the remaining parameters ω, ϕ, and σ^2 are assumed known. Find the maximum likelihood estimator of A.

   (d) Is the estimator for the amplitude A unbiased?

   (e) Suppose the phase ϕ is unknown, but the remaining parameters A, ω, and σ^2 are assumed known. Write the pseudocode for an algorithm that can find the maximum likelihood estimate of ϕ.

5. Consider $X_1, \ldots, X_n$ that is i.i.d. distributed according to a mixture distribution:

$$p(x; \pi) = \pi p_1(x) + (1 - \pi) p_2(x), \quad -\infty < x < \infty$$

   where $0 \leq \pi \leq 1$. Let $\mu_1 = \mathbb{E}_{p_1(x)}[X]$ denote the mean of the first mixture component and $\mu_2 = \mathbb{E}_{p_2(x)}[X]$ denote the mean of the second mixture component. Find the method of moments estimator for π and determine the bias of the estimator.