ESE 531: Statistical Learning and Inference

Homework 3: Numerical Optimization and Evaluation of Estimators

1. Consider a random sample $X_1, \dots, X_n \overset{i.i.d.}{\sim} p(x;\theta)$. Prove whether or not a unique global optimum exists for the likelihood under the following population distributions:

   (a) Gaussian population with unknown mean: $p(x;\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$, where $-\infty < x < \infty$ and $-\infty < \mu < \infty$.

   (b) Gaussian population with unknown precision (inverse variance): $p(x;\alpha) = \sqrt{\frac{\alpha}{2\pi}} \exp\left(-\frac{\alpha}{2}(x-\mu)^2\right)$, where $-\infty < x < \infty$ and $\alpha > 0$. Note, replacing $\alpha = \frac{1}{\sigma^2}$ recovers the standard parameterization for a Gaussian distribution.

   (c) Exponential population: $p(x;\lambda) = \lambda e^{-\lambda x}$, where $x > 0$ and $\lambda > 0$.

   (d) Gamma population: $p(x;\theta) = \frac{1}{\Gamma(k)\theta^k} x^{k-1} \exp\left(\frac{-x}{\theta}\right)$, where $x > 0$ and $\theta > 0$.

2. Time series data can be modeled as autoregressive processes. For example, consider a random sample that following a first-order autoregressive process (i.e., an AR(1) process):

$$X_i = aX_{i-1} + \epsilon_t, \quad i = 1, \dots, n$$

   where $\epsilon_t \sim \mathcal{N}(0,1)$ and $X_0 = x_0$ is assumed to be fixed and known. Here, we say that X_i is conditionally independent of all other X_j given X_{i-1}.

   (a) Write down the log-likelihood of a for observed data $x_1, \dots, x_n$.

   (b) Find the maximum likelihood estimator of a.

   (c) Suppose that X_k is a latent random variable; that is, assume that it cannot be observed and will be missing from the data record. Derive the EM algorithm updates for finding the maximum likelihood estimator.

3. Consider the probabilistic model:

$$X_i = Ar^i + \epsilon_i, \quad i = 1, \dots, n,$$

   where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ is white Gaussian noise with variance σ^2 and $r > 0$ is known.

   (a) Find the CRLB for A.

   (b) Find the maximum likelihood estimator for A and derive its variance.

   (c) What happens to the variance as $n \to \infty$ for various values of r?

4. Consider the probabilistic model:

$$X_i = r^i + \epsilon_i, \quad i = 1, \dots, n,$$

   where $\epsilon_i \sim \mathcal{N}(0, \sigma^2)$ is white Gaussian noise with variance σ^2. Find the CRLB for r.

5. Consider a discrete random sample from a Poisson population, i.e., $X_i \overset{i.i.d.}{\sim} \text{Poisson}(\alpha)$, where:

$$p(x;\alpha) = \frac{\alpha^x}{x!} e^{-\alpha}, \quad x = 0, 1, \dots,$$

   and $\alpha > 0$. Suppose that α is unknown and we would like to estimate it.

   (a) Find the CRLB for α.

   (b) Derive the method of moments estimator of α and compute its variance. Is the estimator efficient?

   (c) Derive the maximum likelihood estimator of α and compute its variance. Is the estimator efficient?