Gaussian Random Variables

$$X \sim N(\mu, \sigma^2)$$

$$\mathbb{E}[X] = \mu \qquad \mathbb{V}[X] = \sigma^2$$

$$Z \sim N(0, 1)$$

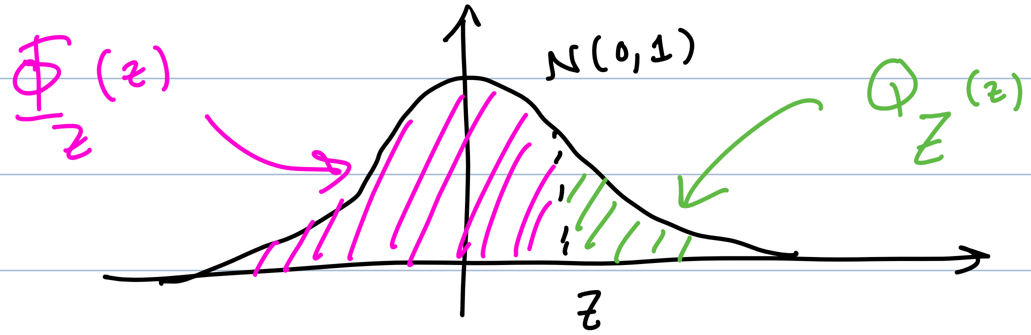
$$\boxed{X = \mu + \sigma Z}$$

→ Cumulative distribution function (CDF)

↳ left-tail probability

$$\Phi_Z(z) = \mathbb{P}(Z \leq z)$$

$$= \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt$$



→ Complementary CDF (Survival function)

$$Q_Z(z) = 1 - \Phi_Z(z) = \mathbb{P}(Z > z)$$

$$= \int_z^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt$$

→ What is the CDF of a non-central Gaussian?

$$X \sim N(\mu, \sigma^2)$$

$$\Phi_X(x) = \mathbb{P}(X \leq x)$$

$$= \mathbb{P}(\mu + \sigma Z \leq x)$$

$$= \mathbb{P}\left(Z \le \frac{x-\mu}{\sigma} \right)$$

$$= \Phi_Z \left(\frac{x-\mu}{\sigma} \right)$$

standardization

$$Q_X(x) = Q_Z\left(\frac{x-\mu}{\sigma} \right)$$

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② Chi-Squared Random Variables

Y is Chi-Squared RV if

$$Y = \sum_{i=1}^{\nu} X_i^2 \sim \chi^2_\nu$$

where X_i are i.i.d. standard normal random variables

PDF: $$p(y) = \frac{1}{2^{\nu/2}\,\Gamma\left(\frac{\nu}{2}\right)}\, y^{\frac{\nu}{2}-1} \exp\left(\frac{-y}{2}\right)$$

$$y \geq 0$$

$\boxed{\text{Chapter 2 (kay)}}$ Detection Theory

$$Q_{\chi^2_\nu}(y) = \begin{cases} 2\,Q_z(\sqrt{y}), & \nu = 1 \\ e^{-\frac{1}{2}y} \displaystyle\sum_{k=0}^{\frac{\nu}{2}-1} \frac{(y/2)^k}{k!}, & \nu \text{ even} \\ \sim\sim\sim, & \nu \geq 3 \text{ and } \nu \text{ odd} \end{cases}$$

Hypothesis testing

$\boxed{\text{Idea}}$ - Given observed data, we want to decide if a signal is present or not!

We phrase this using a binary hypothesis test:

(null hypothesis) $\mathcal{H}_0 : \Theta \in \Theta_0$ — $\mu = 0$, $\mu > 0$, $\sigma^2 = 1$

(alterative hypothesis) $\mathcal{H}_1 : \Theta \notin \Theta_0$ — $\mu \neq 0$, $\mu \leq 0$, $\sigma^2 \neq 1$

<u>Ex:</u> Detection of Aircraft in radar

$$\underline{x} = [x_0, x_1, \ldots, x_T]^T$$

$\mathcal{H}_0$: No aircraft $\Longleftrightarrow$ $\mu = 0$

$\mathcal{H}_1$: There is an aircraft $\Longleftrightarrow$ $\mu \neq 0$

→ $\bar{X} = \frac{1}{T} \sum_{t=1}^{T} X_t$ is an informative statistic for this problem

→ We can naively decide if an aircraft is present by checking if

$$\boxed{\bar{X} \geq \gamma}$$

Thresholding the statistic

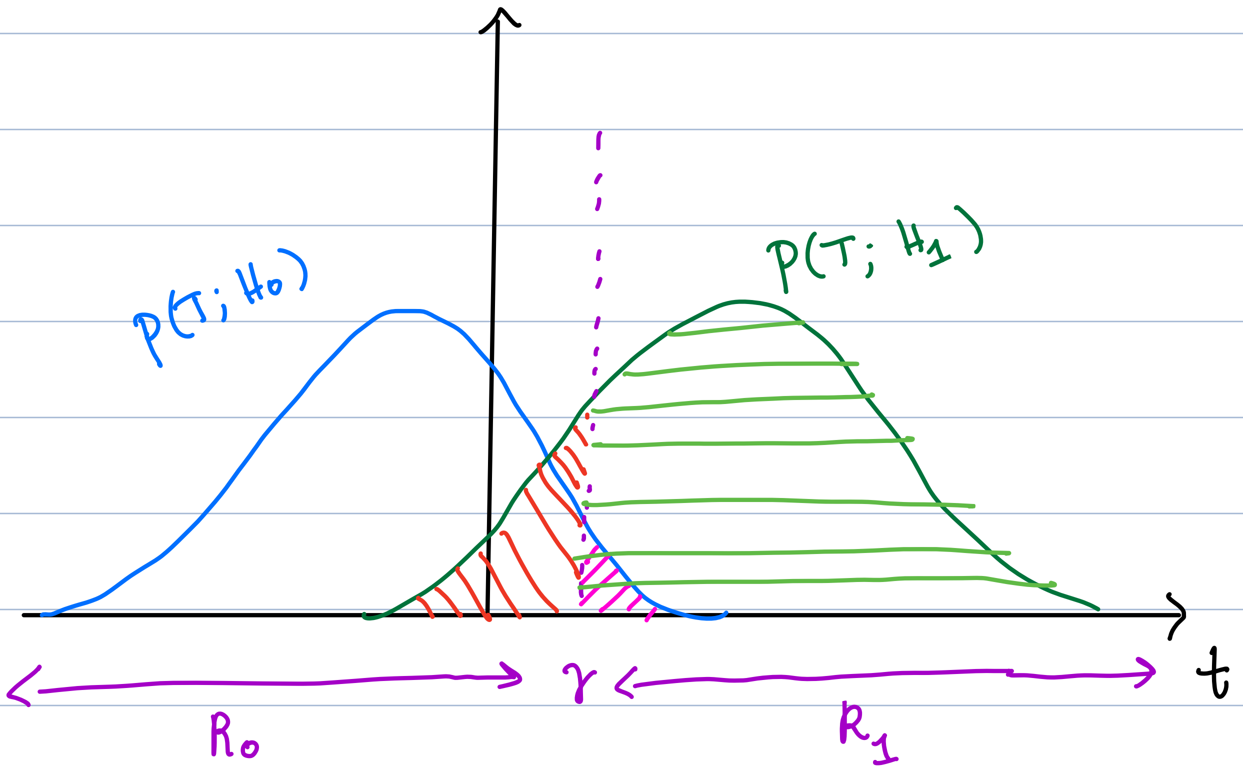
↳ if γ is large, we may never reject the null (type-II error or missed detection)

↳ if γ is too small, we may think the aircraft is almost always present (type-I error or false alarm)

→ Suppose in general we decide $\mathcal{H}_1$ if $T(\underline{x})$ where $T(\cdot)$ is a statistic surpasses some threshold γ:

→ Let $p(T(\underline{x}); \mathcal{H}_0)$ be the distribution of our statistic under the null hypothesis

→ Let $p(T(\underline{x}); \mathcal{H}_1)$ be the distribution of our statistic under the alternative hypothesis:



False alarm probability $P_{FA} = \mathbb{P}(T(x) > \gamma;\, H_0) = \int_{\gamma}^{\infty} p(T(x);\, H_0)\, dt$

Missed detection probability $P_{MD} = \mathbb{P}(T(x) < \gamma;\, H_1) = \int_{-\infty}^{\gamma} p(T(x);\, H_1)\, dt$

probability of detection $P_D = \mathbb{P}(T(x) > \gamma;\, H_1) = \int_{\gamma}^{\infty} p(T(x);\, H_1)\, dt$

Hypothesis test is just an optimization problem
↳ want to find a statistic and threshold such that I maximize P_D subject to $P_{FA} = \alpha$

A <u>detector</u> is function defined as follows:

$$\delta(\underline{x}) = \begin{cases} \text{"Reject null"}, & \overbrace{T(\underline{x}) > \gamma}^{T(\underline{x}) \in R_1} \\ \text{"Retain null"}, & \underbrace{T(\underline{x}) \leq \gamma}_{T(\underline{x}) \in R_0} \end{cases}$$

$\boxed{\text{Neyman - Pearson}}$ To maximize P_D for a given $P_{FA} = \alpha$, decide $\mathcal{H}_1$ if

likelihood ratio

$$L(\underline{x}) \triangleq \boxed{\frac{p(x; \mathcal{H}_1)}{p(x; \mathcal{H}_0)}} > \gamma$$

where the threshold γ is found from:

$$P_{FA} = \int_{\{x : L(x) > \gamma\}} p(x; \mathcal{H}_0)\, dx = \alpha$$

→ This lemma is anagously referred to as the <u>likelihood ratio test</u>

→ <u>COOL THING</u>: If a sufficient statistic for a random sample exists, the likelihood ratio will be a function of the sufficient statistic

<u>Example</u>: $X_i \overset{iid}{\sim} N(\underset{}{\overset{\text{unknown}}{\mu}}, \underset{\text{known}}{\sigma^2})$

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is a sufficient statistic

<u>Proof</u>: We define the Lagrangian:

$$L = P_D + \lambda\left(P_{FA} - \alpha\right) \quad , \quad \lambda < 0$$

$$= \int_{R_1} p(x; \mathcal{H}_1)\,dx + \lambda\left(\int_{R_1} p(x; \mathcal{H}_0)\,dx - \alpha\right)$$

$$= \int_{R_1} \underbrace{p(x;H_1) + \lambda p(x;H_0)}_{\substack{\text{when is}\\ \text{the integrand}\\ \text{is positive}}} dx \; - \lambda \alpha$$

$$p(x;H_1) + \lambda p(x;H_0) > 0$$

$$R_1 = \left\{ \frac{p(x;H_1)}{p(x;H_0)} > -\lambda \;\; \gamma \right\}$$

The Lagrange multiplier can tell us what the threshold should be → can be found directly by solving the constraint

$$P_{FA} = \int_{R_1} p(x;H_0)\, dx = \alpha$$

Example 3.2 - DC Level in White Gaussian Noise (WGN)

We want to consider the following detection problem:

$$\mathcal{H}_0: x_i \sim N(0, \sigma^2), \quad i=1,\dots,n$$

$$\mathcal{H}_1: x_i \sim N(\mu, \sigma^2), \quad i=1,\dots,n$$

Assume μ and σ^2 are known quantities.

To characterize the detector, we should compute the likelihood ratio:

$$L(x_{1:n}) = \frac{p(x_{1:n}; \mathcal{H}_1)}{p(x_{1:n}; \mathcal{H}_0)} = \frac{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(-\frac{(x_i-\mu)^2}{2\sigma^2}\right)}{\prod_{i=1}^{n} \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} \exp\left(\frac{-x_i^2}{2\sigma^2}\right)}$$

$$= \frac{\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right)}{\exp\left(\frac{-1}{2\sigma^2}\sum_{i=1}^{n}x_i^2\right)} > \gamma$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(x_i-\mu)^2 - \sum_{i=1}^{n} x_i^2\right)\right) > \gamma$$

Take the logarithm of both sides

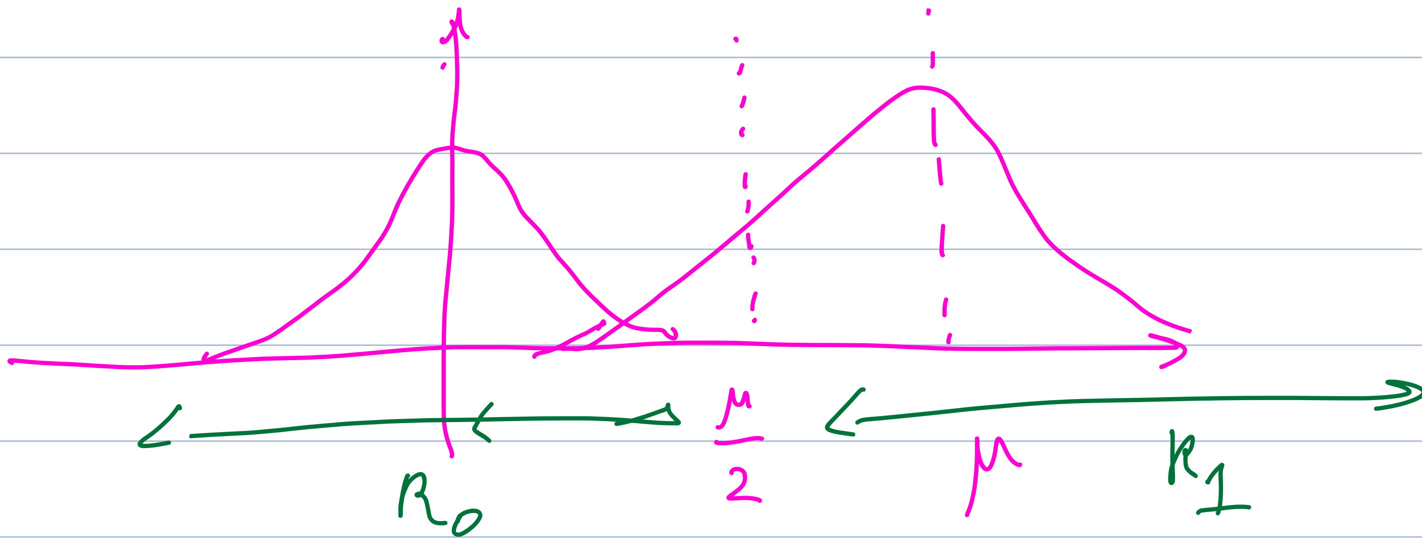
$$-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n} \cancel{x_i^2} - 2x_i\mu + \mu^2 \cancel{- \sum_{i=1}^{n} x_i^2}\right) > \log\gamma$$

$$= \frac{-1}{2\sigma^2}\left(-2\mu\sum_{i=1}^{n} x_i + n\mu^2\right) > \log\gamma$$

$$= \frac{\mu}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{n\mu^2}{2\sigma^2} > \log\gamma$$

$$= \frac{\mu}{\sigma^2}\sum_{i=1}^{n} x_i > \log\gamma + \frac{n\mu^2}{2\sigma^2}$$

$$\Rightarrow \boxed{\frac{1}{n}\sum_{i=1}^{n} x_i > \underbrace{\frac{\sigma^2}{n\mu}\log\gamma + \frac{\mu}{2}}_{\gamma'}}$$



Suppose, I want to set $P_{FA} = \alpha$

our detector :

$$\bar{x} \geq \gamma'$$

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

likelihood of our test statistic under the null

$$P_{FA} = \underbrace{\int_{\gamma'}^{\infty}}_{\text{region } R_1 \text{ (where we reject null)}} p(T(\underline{x}); \mathcal{H}_0)\, dt$$

$$T(\underline{x}) = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \qquad x_i \overset{iid}{\sim} N(0, \sigma^2)$$

$$\bar{x}; H_0 \sim N\left(0, \frac{\sigma^2}{n}\right)$$

$$P_{FA} = \alpha = Q_Z\left(\frac{\gamma'}{\sigma/\sqrt{n}}\right)$$

$$Q_Z^{-1}(\alpha) = \frac{\gamma'}{\sigma/\sqrt{n}} \Rightarrow \boxed{\gamma' = \frac{\sigma\, Q_Z^{-1}(\alpha)}{\sqrt{n}}}$$

Example 2 (3.3 in Detection Book)

$$H_0: X_i \overset{iid}{\sim} N(0, \sigma_0^2) \qquad i = 1, \dots, n$$

$$H_1: X_i \overset{iid}{\sim} N(0, \sigma_1^2) \qquad i = 1, \dots, n$$

$\sigma_1^2 > \sigma_0^2$ both known

$$L(\underline{x}) = \frac{p(x_{1:n}; \mathcal{H}_1)}{p(x_{1:n}; \mathcal{H}_0)} = \frac{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(\frac{-x_i^2}{2\sigma_1^2}\right)}{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(\frac{-x_i^2}{2\sigma_0^2}\right)}$$

$$= \frac{(\cancel{2\pi}\sigma_1^2)^{-n/2}}{(\cancel{2\pi}\sigma_0^2)^{-n/2}} \cdot \frac{\exp\left(\frac{-1}{2\sigma_1^2}\sum_{i=1}^{n} x_i^2\right)}{\exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} x_i^2\right)}$$

$\log(\cdot)$

$$\Rightarrow \; -\frac{n}{\cancel{2}}\log(\sigma_1^2) + \frac{n}{\cancel{2}}\log(\sigma_0^2) - \frac{1}{\cancel{2}\sigma_1^2}\sum_{i=1}^{n} x_i^2 + \frac{1}{\cancel{2}\sigma_0^2}\sum_{i=1}^{n} x_i^2 > 2\gamma$$

$$\Rightarrow \left(\sum_{i=1}^{n} x_i^2\right)\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) > 2\gamma + n\log(\sigma_1^2) - n\log(\sigma_0^2)$$

$$Y = \sum_{i=1}^{n} x_i^2 > \underbrace{\frac{2\gamma + n\log(\sigma_1^2) - n\log(\sigma_0^2)}{\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)}}_{\gamma'}$$

just for intuition

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 > \frac{2\gamma + n\log(\sigma_1^2) - n\log(\sigma_0^2)}{n\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)}$$

$x_i \sim N(0, \sigma_0^2)$ under null

$x_i \sim N(0, \sigma_1^2)$ under alternative

$$P_{FA} = \alpha = \int_{\gamma'}^{\infty} p(y; \mathcal{H}_0)\, dy$$

$$Y' = \frac{1}{\sigma_0^2} \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} \left(\frac{x_i}{\sigma_0} \right)^2$$

$$x_i \sim N(0, \sigma_0^2)$$

$$\frac{x_i}{\sigma_0} \sim N(0, 1)$$

Then $Y' \sim \chi_n^2 \Rightarrow Q_{\chi_n^2}\left(\frac{\gamma'}{\sigma_0^2} \right) = \alpha$

degrees of freedom

$$\gamma' = \sigma_0^2 \, Q_{\chi_n^2}^{-1}(\alpha)$$

Bayesian Approach

$$\mathbb{P}(\text{ERROR}) = \mathbb{P}\Big(\overbrace{\text{reject null}}^{\mathcal{H}_1} \,\Big|\, \overbrace{\text{null is true}}^{\mathcal{H}_0}\Big)\,\mathbb{P}\Big(\overbrace{\text{null is true}}^{\mathcal{H}_0}\Big)$$

$$+ \,\mathbb{P}\Big(\overbrace{\text{retain null}}^{\mathcal{H}_0} \,\Big|\, \overbrace{\text{null is not true}}^{\mathcal{H}_1}\Big)\,\mathbb{P}\Big(\overbrace{\text{null is not true}}^{\mathcal{H}_1}\Big)$$

$$= \mathbb{P}(\mathcal{H}_1 | \mathcal{H}_0)\,\mathbb{P}(\mathcal{H}_0) + \mathbb{P}(\mathcal{H}_0 | \mathcal{H}_1)\,\mathbb{P}(\mathcal{H}_1)$$

Maximum likelihood detector : $P(\mathcal{H}_0) = P(\mathcal{H}_1) = \frac{1}{2}$

$$\frac{p(\underline{x} | \mathcal{H}_1)}{p(\underline{x} | \mathcal{H}_0)} > 1$$

Maximum a posteriori (MAP) detector $P(\mathcal{H}_0) \neq P(\mathcal{H}_1)$

$$\frac{p(\underline{x} \mid \mathcal{H}_1)}{p(\underline{x} \mid \mathcal{H}_0)} > \frac{\mathbb{P}(\mathcal{H}_0)}{\mathbb{P}(\mathcal{H}_1)}$$

Bayes Risk: The Bayes risk $\mathcal{R}$ is defined as:

$$\mathcal{R} = \mathbb{E}[\text{Cost}] = \sum_{i=0}^{1} \sum_{j=0}^{1} C_{ij} \, \mathbb{P}(H_i \mid H_j) \, \mathbb{P}(H_j)$$

Cost matrix $\underline{\underline{C}} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}$

The detecter that minimizes the Bayes risk is to decide $\mathcal{H}_1$ if:

Bayes detector

$$\frac{p(\underline{x} \mid \mathcal{H}_1)}{p(\underline{x} \mid \mathcal{H}_0)} > \frac{(C_{10} - C_{00}) \, \mathbb{P}(H_0)}{(C_{01} - C_{11}) \, \mathbb{P}(H_1)}$$

The reason the Bayes risk perspective is nice is because it is easily extendable to multiple hypothesis testing:

$$R = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} C_{ij} \, \mathbb{P}(H_i \mid H_j) \, \mathbb{P}(H_j)$$

Special case: $C_{ii} = 0$

$C_{ij} = 1$ for all $i \neq j$

Then

$$R = \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \mathbb{P}(H_i \mid H_j) \, \mathbb{P}(H_j)$$

What is the decision rule?

For each possible decision → choose $\mathcal{H}_k$

$$\underset{k=0,\dots,M-1}{} \left[C_k(x) = \sum_{j=0}^{M-1} C_{kj}\, \mathbb{P}(x \mid H_j)\, \mathbb{P}(H_j) \right]$$

Decision rule is to choose the H_k such that C_k is minimized